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Extremal energy properties and construction of stable solutions of the Euler equations

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Certain modifications of the Euler equations of fluid motion lead to systems in which the energy decays or grows monotonically, yet which preserve other dynamically important characteristics of the field. In particular, all topological invariants associated with the vorticity field are preserved. In cases where isolated energy extrema exist, a stable steady flow can be found. In two dimensions, highly constrained by vorticity invariants, it is shown that the modified dynamics will lead to at least one non-trivial stationary, generally stable, solution of the equations of motion from any initial conditions. Numerical implementation of the altered dynamics is straightforward, and thus provides a practical method for finding stable flows. The method is sufficiently general to be of use in other dynamical systems.

Insofar as three-dimensional turbulence is characterized by a cascade of energy, but not topological invariants, from large to small scales, the procedure has direct physical significance. It may be useful as a parameterization of the effects of small unresolved scales on those explicitly resolved in a calculation of turbulent flow.

1. Introduction

Much of what is understood of fluid flows can be traced to the conservation, or sometimes the lack of conservation, of various quantities. For example, an important difference between two- and three-dimensional flow is that in the former case singularities cannot form in finite time, because enstrophy is conserved in two dimensions but not in three. In both two and three dimensions the conservation of energy and the fact that vortex lines are frozen to material curves has led to important stability results (Arnol'd 1965*a*, *b*). These theorems have had particular impact for two-dimensional flows, not because the conditions of stability differ essentially from those for three dimensions, but because solutions can more easily be found in two dimensions.

Following Arnol'd, suppose that the Euler equations describe a flow in an infinite-dimensional phase space, $\mathscr{X}$. This space consists of 'isovortical sheets' such that the vorticity configurations on each sheet can be mapped one to the other by a smooth transformation that conserves the circulation around every material contour. The sheets themselves are infinite-dimensional subspaces of $\mathscr{X}$. Because the equations of motion conserve circulation, the subsequent evolution of an initial condition is confined to the sheet it starts on. In fact, the equations of motion also conserve energy and this further confines the evolution to surfaces of constant energy on a given sheet. These surfaces are also multi-dimensional, in that a specification of the energy on the isovortical sheet does not uniquely specify a trajectory of the flow. A useful idealization, if something of an over simplification, of the space $\mathscr{X}$ is shown

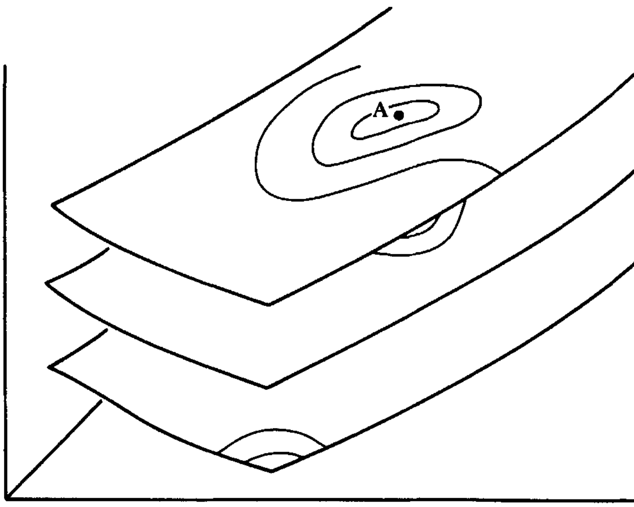

FIGURE 1. Schema of sheets in the fluid phase-space (after Arnol'd 1965*b*). Each sheet is isovortical, meaning that evolution on it conserves circulation. The lines are representations of energy surfaces, of lower dimension, embedded in the isovortical surfaces. Since the evolution of an Euler fluid is confined to a particular energy surface on a given sheet, extremum points (e.g. point A) are stable.

in figure 1 in which the energy surfaces are represented as simple curves on two-dimensional isovortical surfaces.

As Kelvin (1887) appreciated, stationary states are points for which energy is stationary with respect to variations on a given sheet. This was proved by Arnol'd (1965*a*, *b*). Stability of these steady solutions requires analysis of the second variation of energy. It is plausible geometrically that a perturbed system will not necessarily stay close to its parent unless the stationary point is also an extremum of energy (and not a saddle point). However, if the stationary point is an extremum, then the flow is stable in the sense of Lyapunov. That is, the size of the perturbation is bounded by the size of the initial perturbation for all time. This argument, made rigorous by Arnol'd (1965*a*, *b*) forms the physical basis of our procedure.

Some of the preceding remarks need qualification. These are special circumstances in which an energy extremum may not be stable. Suppose, for instance, that for a particular sheet an energy maximum is not a point, but instead is a ring or a line, as in figure 2, or even a plateau. Then any perturbation of the system around the maximum can cause it to move away, and the distance it can move (along an energy contour) is not bounded by the size of the initial perturbation. The ridge may be the only maximum energy value in a given sheet. Thus, even if the energy on the sheet is bounded (from above and below) and the extremum states are in all senses non-trivial, there seems no absolute guarantee that a stable state exists. Mathematically, the second variation of the energy at these extrema is a singular quadratic form, since the variation of energy along the ridge is zero. We shall refer to these cases as singular extrema, and use the unqualified noun for the more generic non-singular, isolated, extrema.

In two-dimensional incompressible flow, these arguments can be used to give simple practical stability criteria because for steady solutions the stream function ψ is functionally related to the vorticity $\zeta = \nabla^2\psi$ by $\psi = g(\zeta)$, where g is any integrable function. Both ζ and $g(\zeta)$ are conserved following material particles. Thus we may

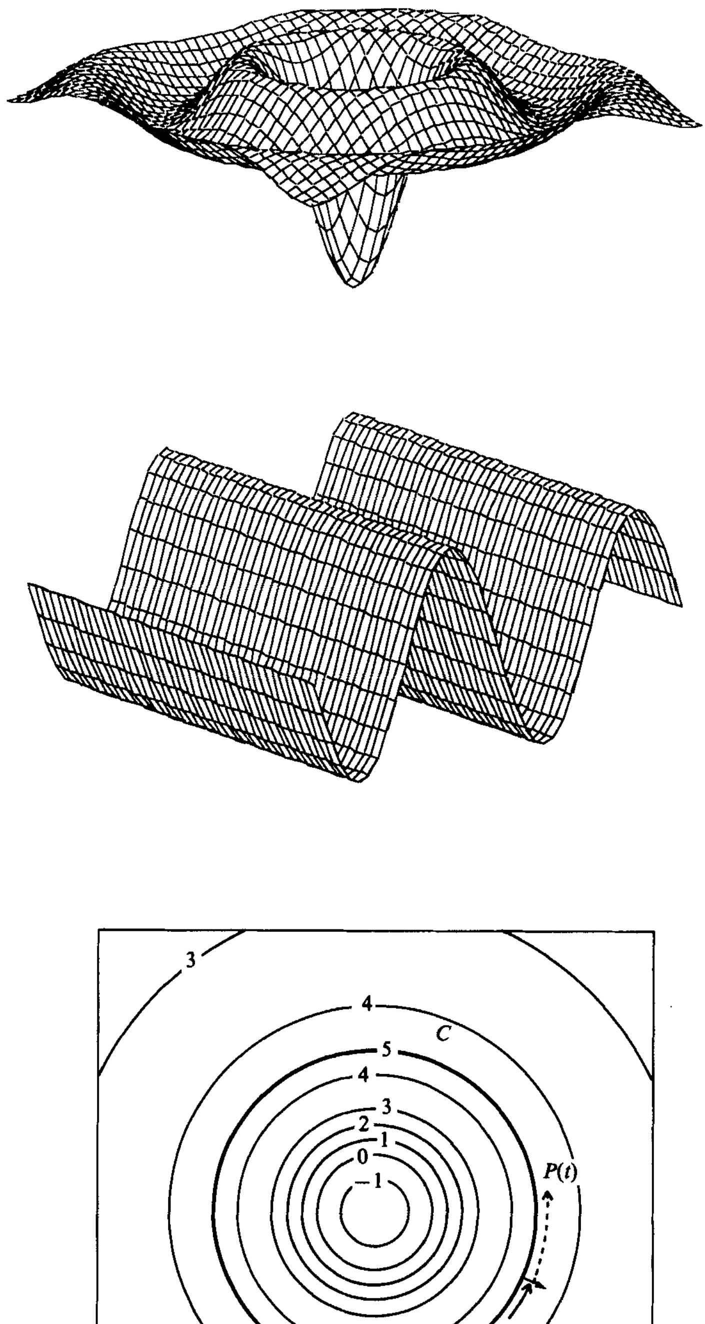

FIGURE 2. (a, b) Idealized three-dimensional plots of energy on a sheet. In both cases energy maxima exist that are not stable. (c) Contour plot of case (a). The growth of a perturbation P off the energy maximum contour C is not bounded by its initial size.

take the variation of the conserved functional $\mathscr{A} = E + \int f(\zeta)\,\mathrm{d}A$, where $f' = g$ and E is the energy. As Kelvin noted, the first variation vanishes. The second variation is

$$\delta^2\mathscr{A} = \frac{1}{2}\int (\nabla\delta\psi)^2 + g'(\zeta)\,(\delta\zeta)^2\,\mathrm{d}A,$$

where the integral is over the two-dimensional domain. If it can be shown that this is of definite sign, stability ensues, because we have shown that the system is at an extremal energy state with respect to other states on the same sheet. Although the form above is unique to two-dimensions, the principle is the same in any number of dimensions, namely that a state is stable if it is an extremal energy state with respect to neighbouring isovortical flows. If the energy extrema are only local, then clearly a large enough perturbation may move the system away from this point and the subsequent growth of the perturbation may not be bounded by its initial size.

The singular, possibly pathological, cases aside, extremum energy states are stable and therefore of great interest. Is it possible to find such states, and indeed do they exist? In three dimensions it seems unlikely that stable solutions of non-trivial topology exist, because the energy on a given isovortical sheet may not be bounded from above or from below (except by zero). On the other hand, in two dimensions many classes of stable stationary states can be generated numerically. (When referring to stability of two-dimensional flow we implicitly mean stability with respect to two-dimensional perturbations only.) Circular vortex patches are an example of a stable flow of relatively simple topology. We show in §6 that in two dimensions on the closure of *any* given sheet extremal energy states exist. Further, we show it is possible to monotonically move towards such states by systematically moving across energy contours while staying on the same isovortical sheet. We do this by advecting the vorticity configuration with an artificial velocity field obtained from a set of 'modified dynamics'. Since the process is still an advection, the mapping will be isovortical and the system stays on the same sheet. The modified dynamics is constructed so that the energy change is guaranteed monotonic, and its steady solutions are the same as those of the real equations of motion. Thus the system must evolve to a stable solution (or to a singular extremum). Surprisingly, it is possible to construct a modified dynamics in both two and three dimensions in a number of ways.

The above philosophy is similar to that of Moffatt (1985), although it differs in a number of important respects. For instance, our procedure preserves vortex topology, whereas Moffatt's preserves streamline topology. Moffatt's procedure finds both stable and unstable solutions, whereas ours will generally ignore saddle points. Because unstable solutions are also of interest, this can be an advantage of Moffatt's method. Similarly, in two dimensions Moffatt's procedure only finds steady solutions (which must of course be of the form $\psi = g(\zeta)$) of certain streamline topology. In contrast our procedure offers a means of finding the steady stable flows which preserve all the vortical invariants of the initial conditions.

In addition to their use as a tool to locate steady flows, the modified dynamics have direct physical significance. In three-dimensional turbulence there is some evidence, and much speculation, that energy, but not the 'topological invariants', is cascaded away from the large scales leaving behind a residual flow now of lower energy. Modified dynamics has a similar effect, suggesting their use as a parameterization of turbulent effects in large-eddy calculations. Because the energy is minimized while helicity and other topological or 'vortical' invariants are conserved, the resulting flow is a generalized 'Beltrami' flow. (A pure Beltrami flow has velocity parallel to vorticity, and arises from minimizing energy while conserving global helicity.)

In the following sections we present a number of recipes for modified dynamics which differ in detail, if not in essence. The first is a general form appropriate for the Euler equations in two or three dimensions, both for incompressible and compressible flow. We examine the conservation properties of the set. Next we specialize to two dimensions. Finally, we present a rather different version appropriate for stratified flow. In this version the modified dynamics may have some physical justification as a parameterization of the radiative effects of gravity waves, in carrying energy away from an isolated region of vorticity. After this we discuss modified magnetohydrodynamics, the existence of non-trivial solutions, and the implications for turbulence.

2. Modified dynamics

We require a set of altered dynamics such that energy is monotonically dissipated or generated. However, the simple use of a viscosity is inappropriate, since we require that the system remains on the same equivortical sheet. This can be done if the system is evolved by some sort of advective process, suggesting we should modify the advecting velocity field. Since we require that the method be isovortical, this modification should be the only change to the vorticity equation.

2.1. *The general form for incompressible flow*

To emphasize the simplicity of the form we shall present the altered dynamics for incompressible flow of constant (unit) density. The generalization to more arbitrary fluids will be given in §3. With an eye to the form of the vorticity equation we write the Euler equations as

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla b, \tag{2.1a}$$

with

$$\nabla \cdot \boldsymbol{u} = 0, \quad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \quad b = p + \tfrac{1}{2}u^2. \tag{2.1b}$$

(The form of b in terms of p (the pressure) and $\boldsymbol{u}$ is actually irrelevant here.) We assume the equations to be valid in a domain $\mathscr{D}$ which may be finite or infinite, but in which there is in any case no contribution to any of the integrals in the following manipulations from the boundaries or from infinity.

Energy conservation follows easily by taking the dot product with $\boldsymbol{u}$ and integrating over $\mathscr{D}$. The nonlinear term vanishes, and the term $\boldsymbol{u} \cdot \nabla b = \nabla \cdot (\boldsymbol{u} b)$ similarly disappears provided there are no boundary contributions. Thus

$$E = \frac{1}{2}\int_{\mathscr{D}} u^2 \,\mathrm{d}V, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = 0.$$

Consider, now, the following set of equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \tilde{\boldsymbol{u}} \times \boldsymbol{\omega} = -\nabla \tilde{b}, \tag{2.2}$$

where

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \alpha \frac{\partial \boldsymbol{u}}{\partial t}, \tag{2.3a}$$

or

$$\tilde{\boldsymbol{u}} = \boldsymbol{u} + \alpha \nabla \times \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}), \tag{2.3b}$$

and $\tilde{b} = p + \frac{1}{2}\tilde{u}^2$. To close the set we add

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \quad \nabla \cdot \tilde{\boldsymbol{u}} = \nabla \cdot \boldsymbol{u} = 0. \tag{2.4}$$

The energetics of the closed set (2.2), (2.3) and (2.4) is obtained by taking the dot product of (2.2) with $\tilde{\boldsymbol{u}}$ for (2.3*a*), or $\boldsymbol{u}$ for (2.3*b*), and integrating over $\mathscr{D}$. The nonlinear terms vanish and it is easily verified that

$$E = \frac{1}{2}\int_{\mathscr{D}} u^2 \,\mathrm{d}V, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\alpha \int_{\mathscr{D}} u_t^2 \,\mathrm{d}V. \tag{2.5a}$$

or

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\alpha \int_{\mathscr{D}} [\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega})]^2 \,\mathrm{d}V. \tag{2.5b}$$

Thus, the energy in the $\boldsymbol{u}$-field monotonically decreases (or increases, depending on the sign of α) until a steady state is reached. Whenever there is unsteady motion, energy changes monotonically. If and when a steady state is reached, then the altered dynamics become identical to the original dynamics for then the extra terms vanish. In particular, steady solutions of (2.2) satisfy

$$\boldsymbol{u}\times\boldsymbol{\omega} = -\boldsymbol{\nabla} b,$$

along with the constraints $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}, \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. Since whenever there is motion energy monotonically decreases (increases), the fluid must either tend towards a state of rest or infinite energy (but with, as will be shown below, conserved circulation, helicity and potential vorticity) or to a non-trivial solution of the Euler equations.

Now, the modified vorticity equation is obtained by taking the curl of (2.2). This gives

$$\frac{\partial\boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\tilde{\boldsymbol{u}}\times\boldsymbol{\omega}) = 0, \tag{2.6a}$$

or

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\tilde{\boldsymbol{u}} = 0. \tag{2.6b}$$

These equations are revealing, since they have the same form as the conventional vorticity equation, except for the use of modified velocity. In other words, the vorticity is being evolved, not by the 'true' velocity, but nevertheless through the convective action of a velocity field. The lines of vorticity are frozen into the modified fluid. Thus, we expect circulation and other properties to be conserved, and this is explicitly verified below. Hence the mapping from initial to final state is isovortical: if the initial state is on a particular sheet $\mathscr{S}$ in $\mathscr{K}$, then even under modified dynamics it stays on that sheet. From the discussion in the introduction, if a non-trivial final state exists, it will be stable except for the special cases mentioned.

3. Conservation properties

3.1. *Circulation*

The circulation around a closed material line frozen in the fluid is

$$\mathscr{C} = \oint \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{r},$$

where the path of the integral follows the same fluid at all times. It is a well-known conserved quantity for the usual dynamics. It is conserved by the modified dynamics because the vorticity equation has the same form as the equation for the relative displacement of neighbouring material parcels $\delta\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, namely

$$\frac{\partial\delta\boldsymbol{x}}{\partial t} + (\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\,\delta\boldsymbol{x} - (\delta\boldsymbol{x}\cdot\boldsymbol{\nabla})\,\tilde{\boldsymbol{u}} = 0.$$

Thus, a vortex tube moves materially with the modified fluid, and in particular the integral of vorticity across a material surface will be invariant.

Explicitly, by Stokes theorem the circulation is

$$\mathscr{C} = \int \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}. \tag{3.1}$$

Considering small material elements $\delta \boldsymbol{S}$ of $\boldsymbol{S}$, moving with the fluid, the rate of change of circulation of each is given by

$$\begin{aligned}\frac{\mathrm{d}\delta\mathscr{C}}{\mathrm{d}t} &= \left[\frac{\tilde{\mathrm{D}}\boldsymbol{\omega}}{\mathrm{D}t} \cdot \delta \boldsymbol{S} + \boldsymbol{\omega} \cdot \frac{\mathrm{d}\delta \boldsymbol{S}}{\mathrm{d}t}\right] \\ &= [\omega_i \partial_i \tilde{u}_j \delta S_j - \delta S_j \partial_i \tilde{u}_j \omega_i] \\ &= 0,\end{aligned} \tag{3.2}$$

where we have used (see e.g. Batchelor 1967)

$$\frac{\mathrm{d}\delta S_i}{\mathrm{d}t} = -\delta S_j \frac{\partial \tilde{u}_j}{\partial x_i} + O(|\delta \boldsymbol{S}|^2)$$

for a constant-density fluid, and defined

$$\frac{\tilde{\mathrm{D}}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}.$$

Hence, circulation is conserved in the modified fluid, by which we mean that the integral of unmodified velocity around a material surface is constant, provided by material surface we mean marked parcels advected by $\tilde{\boldsymbol{u}}$.

3.2. *Helicity*

Helicity is a measure of the degree of knottedness of a vector field, such as a vorticity field (Moffatt 1969). It is defined by

$$\mathscr{H} = \int \boldsymbol{u} \cdot \boldsymbol{\omega} \, \mathrm{d}V, \tag{3.3}$$

where the integral is over a volume V enclosing and moving with the fluid. It is fairly straightforward to show that it is conserved by the Euler equations with certain restrictions on the domain. We now show this is also true when the velocity and vorticity fields are being evolved by the modified velocity, namely (2.2) and (2.3).

To obtain an equation for the helicity, take the dot product of (2.2) with $\boldsymbol{\omega}$ and take the dot product of (2.6) with $\boldsymbol{u}$ to give

$$\frac{\tilde{\mathrm{D}}(\boldsymbol{u} \cdot \boldsymbol{\omega})}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}(\boldsymbol{u} \cdot \tilde{\boldsymbol{u}} - b).$$

Now,

$$\begin{aligned}\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} &= \int \frac{\tilde{\mathrm{D}}(\boldsymbol{u} \cdot \boldsymbol{\omega})}{\mathrm{D}t} \mathrm{d}V \\ &= \int \boldsymbol{\omega} \cdot \boldsymbol{\nabla}(\boldsymbol{u} \cdot \tilde{\boldsymbol{u}} - b) \, \mathrm{d}V \\ &= \int (\boldsymbol{u} \cdot \tilde{\boldsymbol{u}} - b) \, \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}\end{aligned}$$

using the divergence theorem. Thus, $\mathrm{d}\mathscr{H}/\mathrm{d}t = 0$ and helicity is conserved under the altered dynamics, provided that the volume V over which the integral is taken moves with the modified velocity and the component of vorticity is zero normal to the bounding surface, or the domain is infinite and the vorticity falls away sufficiently quickly.

3.3. *Potential vorticity*

At this point it is convenient to generalize our formalism to consider compressible flows. For a general Euler fluid, we write the modified dynamics as

$$\frac{\partial \boldsymbol{u}}{\partial t}-\tilde{\boldsymbol{u}}\times\boldsymbol{\omega}=-\frac{1}{\rho}\nabla p-\tfrac{1}{2}\nabla\tilde{u}^2, \tag{3.4}$$

$$\frac{\tilde{\mathrm{D}}\rho}{\mathrm{D}t}+\rho\nabla\cdot\tilde{\boldsymbol{u}}=0, \tag{3.5}$$

where

$$\frac{\tilde{\mathrm{D}}}{\mathrm{D}t}=\frac{\partial}{\partial t}+\tilde{\boldsymbol{u}}\cdot\nabla. \tag{3.6}$$

The equations may be closed by the addition of an equation of state and (if the fluid is not barotropic) an internal energy equation. The essential point about these equations is that with the exception of velocity all quantities, including density and passive vector and scalar tracers, are simply advected by the modified velocity field $\tilde{\boldsymbol{u}}$ which is related to $\boldsymbol{u}$ by (2.3). (The exception to this rule is (3.4), which (because $\boldsymbol{\omega}$ is not passive) is not simply a velocity equation with the advection terms using the modified velocity. If such a scheme were used other conservation quantities would be lost.) Note that the energy of the system will still be obtained by multiplying (3.4) by $\tilde{\boldsymbol{u}}$ or $\boldsymbol{u}$ and integrating over $\mathscr{D}$. From (3.4) the extra, negative (or positive) definite term in the energy equation $\int \alpha u_t^2\,\mathrm{d}V$, or $\int \alpha[\nabla\times\boldsymbol{u}\times\boldsymbol{\omega}]^2\,\mathrm{d}V$, arises. However, all other terms vanish since they have a one-to-one correspondence with similar terms in the unmodified Euler equations, obtained by $\tilde{\boldsymbol{u}}\rightarrow\boldsymbol{u}$. In short, for any fluid with any equation of state the energy-conserving Euler equations may be converted to an energy-dissipating or generating system by the replacement $\boldsymbol{u}\rightarrow\tilde{\boldsymbol{u}}$, where $\tilde{\boldsymbol{u}}$ is given by (2.3), in the nonlinear terms of the momentum equation written in the form (3.4) and in the conservation equations for mass and energy.

From (3.4) we form a vorticity equation:

$$\frac{\partial\boldsymbol{\omega}}{\partial t}-\nabla\times(\tilde{\boldsymbol{u}}\times\boldsymbol{\omega})=-\nabla\times(\nabla p/\rho), \tag{3.7}$$

or

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+(\tilde{\boldsymbol{u}}\cdot\nabla)\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\nabla)\tilde{\boldsymbol{u}}+\boldsymbol{\omega}(\nabla\cdot\tilde{\boldsymbol{u}})=\frac{1}{\rho^2}(\nabla p\times\nabla\rho). \tag{3.8}$$

Again they are in just the same form as the conventional vorticity equation, except that $\boldsymbol{u}$ is replaced by $\tilde{\boldsymbol{u}}$. Thus, the effects of advection are still to distort the vortex lines of the initial state. Lastly, a scalar θ which obeys the unmodified equation $\mathrm{D}\theta/\mathrm{D}t=0$ now obeys

$$\frac{\tilde{\mathrm{D}}\theta}{\mathrm{D}t}=0. \tag{3.9}$$

Now, (3.5), (3.7) or (3.8), and (3.9) are in precisely the same form as conventional equations for vorticity, density and scalar fluid property θ. It is just these equations from which conservation of potential vorticity is derived; hence it is expected that potential vorticity (or any topological invariant) will be conserved by the altered set. To demonstrate this explicitly we first eliminate $\nabla\cdot\tilde{\boldsymbol{u}}$ from (3.8) using (3.5) to give

$$\frac{\tilde{\mathrm{D}}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right)=\left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\tilde{\boldsymbol{u}}+\frac{1}{\rho^3}(\nabla\rho\times\nabla p). \tag{3.10}$$

Now take the dot product of $\nabla\theta$ with (3.10), and add the result to the following relation, which is a consequence of (3.9):

$$\frac{\boldsymbol{\omega}}{\rho}\cdot\frac{\tilde{\mathrm{D}}}{\mathrm{D}t}\nabla\theta = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\frac{\mathrm{D}\theta}{\mathrm{D}t} - \left[\left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\tilde{\boldsymbol{u}}\right]\cdot\nabla\theta,$$

to give

$$\frac{\tilde{\mathrm{D}}}{\mathrm{D}t}\left[\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\theta\right] = \nabla\theta\cdot\left[\frac{\nabla p\times\nabla\rho}{\rho^3}\right]. \tag{3.11}$$

If the fluid is barotropic ($\nabla p\times\nabla\rho = 0$) or θ is a function of p and ρ only, then

$$\frac{\tilde{\mathrm{D}}Q}{\mathrm{D}t} = 0, \tag{3.12}$$

where $Q = (\boldsymbol{\omega}/\rho)\cdot\nabla\theta$ and $\tilde{\mathrm{D}}/\mathrm{D}t$ is defined by (3.6). That is to say, the potential vorticity of a parcel is conserved as it moves with the modified velocity field.

4. Two-dimensional and other special cases

4.1. *Two-dimensional dynamics*

Because of the generality of the prescription (2.2) and (2.3) we should expect no difficulty treating cases other than the three-dimensional one. In particular the two-dimensional case can readily be derived. In the momentum equations for a compressible fluid, the modified two-dimensional equations are just the same as the three-dimensional forms (e.g. (3.4)–(3.6)) except that all spatial derivatives are two-dimensional. The special case of two-dimensional incompressible flow is interesting, since some new results become available. The two-dimensional vorticity equation, from (3.8) is just

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\tilde{\boldsymbol{u}}\cdot\nabla)\,\boldsymbol{\omega} = 0. \tag{4.1}$$

The continuity equation is satisfied by writing the two-dimensional velocity in terms of a stream function in the usual way, and using $\boldsymbol{\omega} = \zeta\hat{\boldsymbol{z}}$ for two-dimensional vorticity we have

$$\frac{\partial\zeta}{\partial t} + J(\tilde{\psi}, \zeta) = 0, \tag{4.2}$$

where

$$\zeta = \nabla^2\psi, \quad \tilde{\psi} = \psi + \alpha\psi_t$$

and $J(a,b) = \partial_x a\,\partial_y b - \partial_y a\,\partial_x b$. The form of the equations is $\tilde{\mathrm{D}}\zeta/\mathrm{D}t = 0$, so vorticity is conserved on parcels, so long as the advecting velocity field is defined as $\tilde{\boldsymbol{u}} = (-\partial_y\tilde{\psi}, \partial_x\tilde{\psi})$. Thus, initial and final state enstrophies (or indeed any integral of any function of vorticity) are the same. If we define an energy by $E = \frac{1}{2}\int(\nabla\psi)^2$ and form the energy equation by multiplying (4.1) by $-\tilde{\psi}$ and integrating over the domain, then

$$-\int\nabla^2\psi_t(\psi + \alpha\psi_t) = 0,$$

so

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \alpha\int\psi_t\nabla^2\psi_t\,\mathrm{d}A = -\alpha\int\nabla\psi_t\cdot\nabla\psi_t\,\mathrm{d}A$$

which is of definite sign. If and when the system achieves a steady state, then $\zeta_t = \psi_t = 0$ and the dynamics are conventional.

The above scheme again should pose no difficulty, in principle, for straightforward numerical implementation. However, the double appearance of a time derivative either necessitates an iterative approach or analytically a diagnostic relationship between ψ, $\tilde{\psi}$ and $J(\psi, \zeta)$, which may be hard to enforce. A poor man's alternative is just to use a prior value of ζ_t to evaluate ψ_t in (4.2): since the set (4.2) may be conveniently condensed into the single equation,

$$\frac{\partial \zeta}{\partial t} + J(\psi + \alpha \psi_t, \zeta) = 0,$$

where $\zeta = \nabla^2 \psi$, the time derivative within the Jacobian is evaluated either at a previous time or in a subsidiary time step.

It is evident that the above form is not unique. A more general form is

$$\frac{\partial \zeta}{\partial t} + J(\psi + \alpha \nabla^{2n} \psi_t, \zeta) = 0. \tag{4.3}$$

It is straightforward to show that this monotonically generates or conserves energy, depending on the power n and the sign of α.

4.2. *A simpler scheme*

For incompressible two-dimensional dynamics, a scheme analogous to (2.3*b*) is available which does not require a second time derivative. Instead of using a time derivative within the Jacobian term of (4.3), we may use the Jacobian of ψ and ζ itself. This is a simple and most straightforward prescription, and certainly easy to implement numerically. Thus,

$$\frac{\partial \zeta}{\partial t} + J(\tilde{\psi}, \zeta) = 0, \quad \tilde{\psi} = \psi + \alpha \nabla^{2n} J(\psi, \zeta)$$

and as usual $\zeta = \nabla^2 \psi$. The set may usefully be condensed into the single equation

$$\frac{\tilde{\mathrm{D}} \zeta}{\mathrm{D}t} = \frac{\partial \zeta}{\partial t} + J(\psi + \alpha \nabla^{2n} J(\psi, \zeta), \zeta) = 0. \tag{4.4}$$

The energy is $E = \int \frac{1}{2} (\nabla \psi)^2 \, \mathrm{d}A$. To obtain the energy equation, multiply (4.4) by $-\psi$ to obtain

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \int \psi J(\tilde{\psi}, \zeta) \, \mathrm{d}A \\ &= \int \psi J(\psi + \alpha \nabla^{2n} J(\psi, \zeta), \zeta) \, \mathrm{d}A \\ &= \alpha \int J(\psi, \zeta) \, \nabla^{2n} J(\psi, \zeta) \, \mathrm{d}A, \end{aligned}$$

which is again of definite sign. It is clear that the energy is a monotonically decreasing or increasing function of time, unless ζ and ψ are functionally related, which is precisely the condition that the system be in a steady state.

4.3. *Quasi-geostrophic dynamics*

Quasi-geostrophic dynamics follow most easily as a simple extension of the two-dimensional case. The governing equations for modified quasi-geostrophic dynamics are

$$\frac{\partial q}{\partial t} + J(\tilde{\psi}, q) = 0, \tag{4.5}$$

where, for example, either

$$\tilde{\psi} = \psi + \alpha\nabla^{2n}J(\psi, q),$$

or

$$\tilde{\psi} = \psi + \alpha\nabla^{2n}q_t,$$

and $q(x, y, z)$ is the potential vorticity given by

$$q = \nabla^2\psi(x, y, z) + \partial_z \lambda^2 \partial_z \psi(x, y, z).$$

Here, $\lambda(z)$ is related to the basic stratification.

The energetics of this set follow by multiplying by (4.5) by $-\psi$ and integrating over the three-dimensional domain $\mathscr{D}$. Assuming essentially conventional boundary conditions at the top and bottom (namely $\tilde{\mathrm{D}}(\partial_z\psi)/\mathrm{D}t = 0$) then it is straightforward to show that

$$E = \frac{1}{2}\int_{\mathscr{D}} (\nabla\psi)^2 + \lambda(z)^2\psi_z^2\,\mathrm{d}V, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\alpha\int_{\mathscr{D}} (J(\psi, q))^2\,\mathrm{d}V. \tag{4.6}$$

It is clear from (4.5) that potential vorticity is still conserved on parcels. A consequence of this is that any function of q, say $G(q)$, is conserved when integrated over the domain. Explicitly

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{D}} G(q)\,\mathrm{d}V = 0 \tag{4.7}$$

and in particular the enstrophy, $Z = \frac{1}{2}\int_{\mathscr{D}} q^2$, is conserved.

4.4. *Stratified flow and the shallow-water equations*

In this subsection we present a variant particularly suited for the shallow-water equations. It is not as general as the above forms, since it cannot be applied to incompressible flow. However, we present it to demonstrate the plurality of possibilities, and because it may have useful applications, perhaps as a physically based parameterization of gravity waves in carrying energy away from a region of activity.

Let us write a modified set of shallow-water equations in a rotating frame of reference as

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{k} \times f\boldsymbol{u} = -g\nabla h - \alpha g\nabla h_t, \tag{4.8}$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{u} = 0. \tag{4.9}$$

The extra term now is the last term on the right-hand side of (4.8); the velocity term is not modified. It is straightforward to show that

$$E = \frac{1}{2}\int hu^2 + gh^2\,\mathrm{d}A, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\alpha\int h_t^2\,\mathrm{d}A,$$

so that the energy monotonically decays or grows. Since the extra term vanishes in the vorticity equation, the potential vorticity is still conserved. This scheme might have application as a parameterization of gravity wave activity, since the modified dynamics remove energy in a region of vortex activity. (Indeed in general circulation models of the atmosphere it is common to use energy-dissipating but enstrophy-conserving schemes.) The analogy with the geostrophic adjustment problem is striking – gravity waves radiate away from a vorticity anomaly, reducing the local energy but not the potential vorticity.

A continuously stratified generalization is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\hat{\boldsymbol{z}}\times f\boldsymbol{u} = -\boldsymbol{\nabla}p-\alpha\boldsymbol{\nabla}\rho_t-g\hat{\boldsymbol{z}}, \tag{4.10}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{4.11}$$

With no modification, $\alpha = 0$, the energy is conserved. For the modified set, the energy balance is

$$E = \frac{1}{2}\int \rho u^2+g\rho z\,\mathrm{d}V, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\alpha\int \rho_t^2\,\mathrm{d}V,$$

so again energy decays until a steady solution is reached.

4.5. *A numerical example*

Here we present an example of simulation of modified dynamics that demonstrates the feasibility of numerical implementation of this scheme. For conceptual simplicity we simulate the evolution of a patch of constant vorticity in a two-dimensional flow. The specific algorithm employed is that given by (4.4) with $n = 0$ and doubly periodic boundary conditions. The numerical scheme is spectral and so for $\alpha = 0$ energy and enstrophy can be conserved as accurately as desired by the choice of a sufficiently small time step. Our (manifestly unstable) initial condition for this example was created by advecting an elliptical vortex patch with a fixed randomly generated velocity field. At infinite resolution, the modified dynamics with energy increasing must take an isolated irregular patch of constant vorticity to a circular patch with the same vorticity and of the same area. We can see evolution towards the axisymmetric state occurring in the top panels of figure 3 where a contour of constant vorticity at the edge of the patch is followed in time. Of course in the simulation the resolution is finite (here equivalent to a 128×128 grid) and consequently patches of constant vorticity cannot actually be realized in detail nor can the infinity of temporal invariants be exactly maintained during the simulation.

In the plot of relative vorticity *vs.* stream function for the initial condition we see that the effect of the finite resolution is to make the step function of relative vorticity values 'fuzzy'. The patch as numerically realized actually has many fluctuations about the interior and exterior constant values. As energy is added isovortically to the flow the fluid particles would tend to arrange themselves in concentric rings of constant vorticity with the highest values in the centre. The scatter plots of relative vorticity *vs.* stream function show the detailed rearrangement of vorticity values that eventually results in a functional relationship between ζ and ψ – a monotonic fall off of ζ with increasing ψ. (Note for the circular patch that ψ increases monotonically with distance from the centre.) During the course of the simulation the energy increases by 22% while the time step used here is small enough to keep the overall change in the enstrophy within 0.1% and most of that change occurs in the very earliest phase of the evolution (see figure 4). At finite resolution, ideal integral vorticity invariants of the form $\int\zeta^p$ will be conserved only if $p = 2$. For $p = \frac{2}{5}, \frac{4}{3}$, 3, 4, 8 we found overall variations of 0.9%, 0.5%, −0.1%, 0.4%, 4.0% respectively. Evidently the influence of the approximate conservation of vorticity invariants, other than enstrophy, is sufficiently strong to prevent the flow from going towards the maximum energy state constrained only by the enstrophy. If enstrophy conservation were the only restriction, then a continual energy increase would evolve

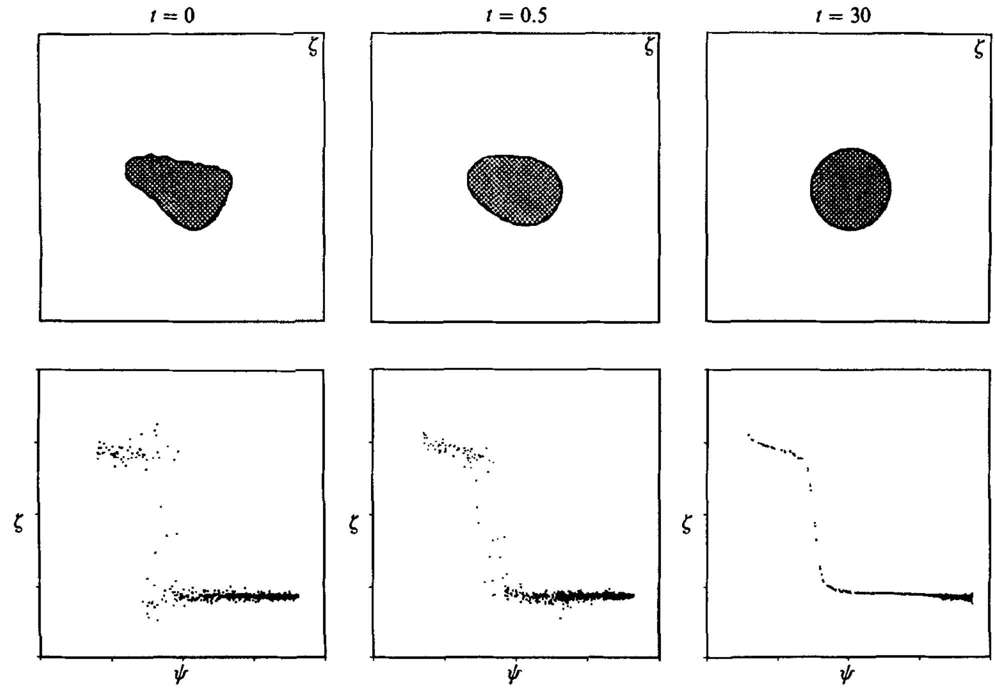

FIGURE 3. Modified dynamics of patch of constant vorticity. The upper row shows the evolution of the vortex patch from the initial irregular structure to the final axisymmetric maximum energy state (only the $\zeta = 0.55$ contour is drawn). The scatter plots of the lower row give the detailed distribution of relative vorticity (resolution 128×128, $\alpha = 0.1$).

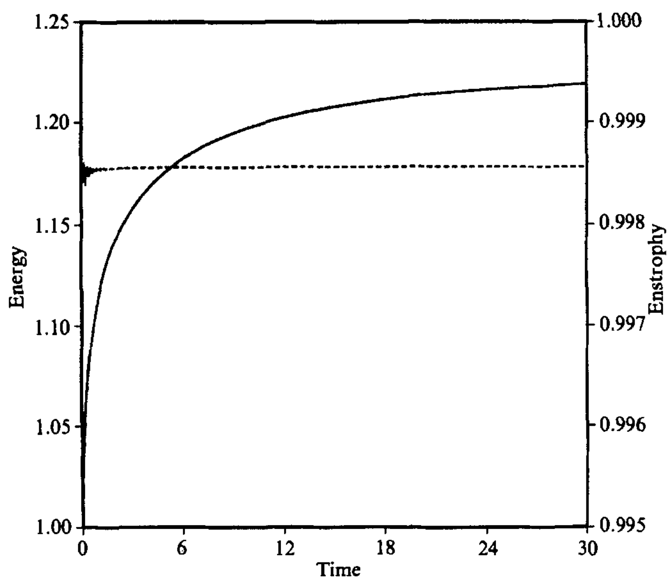

FIGURE 4. Energy and enstrophy evolution for the patch simulation. The energy is shown as a solid curve and the enstrophy as a dashed curve.

the flow towards a state composed of only the largest modes in the periodic domain. In non-dimensional units the maximum possible energy constrained only by the enstrophy is equal to the enstrophy, which for this initial condition is 2.91 times the initial energy. Since the energy in this simulation is levelling off at only 1.22 times its initial value, the effect of the other invariants is clearly demonstrated.

The example shows that the modified dynamics certainly can be integrated, in two dimensions at least, to produce a stable steady flow in finite time without the production of singularities.

5. Magnetohydrodynamics and topological accessibility

5.1. *Moffatt's procedure*

An interesting procedure has been suggested by Moffatt (1985) using the well-known analogy between steady solutions of the Euler equations and the magnetostatic equations. Since there are both similarities and important differences between his method and ours, we shall briefly describe his. Start with the Euler equations for a neutral fluid of constant density, namely

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} - \nabla b,$$

where $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. Steady solutions satisfy

$$\boldsymbol{u} \times \boldsymbol{\omega} = \nabla b, \tag{5.1}$$

and $\nabla \cdot \boldsymbol{u} = 0$. Now, steady, zero-velocity solutions of the magnetohydrodynamic equations (i.e. the magnetostatic equations) satisfy

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla p \tag{5.2}$$

and $\nabla \cdot \boldsymbol{B} = 0$. This obvious correspondence suggests the association

$$\boldsymbol{u} \Leftrightarrow \boldsymbol{B}, \quad \boldsymbol{\omega} \Leftrightarrow \boldsymbol{j}, \quad b \Leftrightarrow -p. \tag{5.3}$$

The viscous magnetohydrodynamic equations (but with zero magnetic resistivity) may be written

$$\left.\begin{aligned} \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} &= -\rho^{-1}\nabla p + \boldsymbol{j} \times \boldsymbol{B} + \mu \nabla^2 \boldsymbol{v}, \\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \\ \boldsymbol{j} &= \nabla \times \boldsymbol{B}, \\ \nabla \cdot \boldsymbol{v} = 0, &\quad \nabla \cdot \boldsymbol{B} = 0. \end{aligned}\right\} \tag{5.4}$$

These equations monotonically lose energy. Explicitly,

$$E = \frac{1}{2}\int (B^2 + v^2)\,\mathrm{d}V, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\mu \int (\nabla \times \boldsymbol{v})^2\,\mathrm{d}V. \tag{5.5}$$

Generally, as long as $\boldsymbol{v} \neq 0$ energy will decrease. A steady state will be achieved when $\boldsymbol{v} = 0$ and (5.2) is satisfied. Thus, suppose we begin with some neutral flow, generally unsteady, characterized by the fields $(\boldsymbol{\omega}^0, \boldsymbol{u}^0, b^0)$. We associate the fields $(\boldsymbol{j}, \boldsymbol{B}, p)$, respectively, with them and use these values as initial conditions in an integration of (5.4), with any choice of the field $\boldsymbol{v}$. The magnetohydrodynamic field evolves, dissipating energy, until (5.2) is satisfied. We now reassociate the fields $(\boldsymbol{j}, \boldsymbol{B}, p)$ with $(\boldsymbol{\omega}, \boldsymbol{u}, b)$ to obtain a solution of the Euler equations, denoted $\boldsymbol{u}^{\mathrm{E}}$.

If 'topological accessibility' is defined to mean accessibility through the advective action of a smooth field $\boldsymbol{v}(x,t)$, then this procedure shows that there is at least one steady Euler flow that is topologically accessible from any (smooth) initial flow. However, this does not imply topological equivalence, since there may be reconnection of streamlines after infinite time. Nor does it imply dynamical accessibility, which would imply that the vortex lines of $\boldsymbol{u}^0$ are deformable to the vortex lines of $\boldsymbol{u}^{\mathrm{E}}$. Thus, the procedure does not guarantee that the final state will be stable. On the other hand, it does guarantee that non-trivial solutions can be found. This is because, if the topology of the initial $\boldsymbol{B}$-field is non-trivial, then there is a minimum magnetic energy $\frac{1}{2}\int B^2$ below which the system cannot fall by any mapping (e.g. advection) which preserves the topology of the field, since lines of magnetic force cannot be cut.

5.2. *Modified magnetohydrodynamics*

It is straightforward to set up a set of modified dynamics for the magnetohydrodynamic equations. These are

$$\left.\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} - \tilde{\boldsymbol{u}} \times \boldsymbol{\omega} &= -\nabla b + \boldsymbol{j} \times \boldsymbol{B},\\ \frac{\partial \boldsymbol{B}}{\partial t} &= \nabla \times (\tilde{\boldsymbol{u}} \times \boldsymbol{B}),\\ \boldsymbol{j} = \nabla \times \boldsymbol{B}, &\quad \boldsymbol{\omega} = \nabla \times \boldsymbol{u},\\ \nabla \cdot \tilde{\boldsymbol{u}} = 0, &\quad \nabla \cdot \boldsymbol{B} = 0.\end{aligned}\right\} \tag{5.6}$$

These equations are similar to the set (5.4), except that (5.6) are 'inviscid', and the velocity field $\tilde{\boldsymbol{u}}$ is the modified velocity (2.3). The unmodified equations conserve energy exactly; the modified equations monotonically dissipate or generate energy according as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int \alpha u_t^2 \,\mathrm{d}V, \quad \text{or} \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\int \alpha [\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega})]^2 \,\mathrm{d}V.$$

The system must either tend towards a solution to the complete set (with non-zero velocity) or to a solution of the magnetostatic equations. The modified set maintains all the topological invariants of the unmodified set, including magnetic and cross-helicity.

6. Structure of solutions and implications for turbulence

6.1. *Two-dimensional flows*

In the two-dimensional case we can prove that interesting solutions (i.e. flow with non-zero and non-infinite energy and with no generation of singularities in the vorticity field) exist. First consider the set where α is chosen so that energy increases. Nevertheless, enstrophy is still conserved. That is

$$Q = \frac{1}{2}\int_{\mathscr{D}} q^2 \,\mathrm{d}A = \frac{1}{2}\int_{\mathscr{D}} (\nabla^2 \psi)^2 \,\mathrm{d}A$$

is invariant, where $\mathscr{D}$ is our domain. Now, energy E is given by

$$E = \frac{1}{2}\int_{\mathscr{D}} u^2 \,\mathrm{d}A = \frac{1}{2}\int_{\mathscr{D}} (\nabla\psi)^2 \,\mathrm{d}A.$$

Although it monotonically increases until a solution is reached, its value is bounded from above. This follows from Poincaré's inequality

$$\int (\nabla^2\psi)^2 \,\mathrm{d}A \geqslant C \int (\nabla\psi)^2 \,\mathrm{d}A, \tag{6.1}$$

where C is some constant and $\nabla\psi$ vanishes on the boundary of the integral. Because the left-hand side is constant the energy is bounded from above.

Now, modified dynamics rearranges the initial vorticity field, always increasing the energy. But because of the inequality above, this process must eventually cease. We are thus led to the following remarkable conclusions. For any two-dimensional flow, there exists at least one stationary solution of the Euler equations accessible by a rearrangement of the vorticity field. Excepting special cases, this state will be stable. Furthermore, a velocity field that will advect the vorticity field to a solution from any initial flow can be found. The method certainly does not enable every stable solution on a given isovortical sheet to be found, even if there exists more than one. Also, the method may take an infinite time to reach a solution, especially if the solution is at the 'edge' of a sheet. In that case vorticity reconnection can occur.

The case where the energy decreases is less clear cut. For instance it is possible for the energy to become arbitrarily small while all the vorticity invariants are conserved. This is possible if the domain contains no net vorticity, i.e. $\int \zeta \,\mathrm{d}A = 0$. In this case the vorticity can be teased out into infinitely fine filaments, so that in any finite subdomain the net vorticity is arbitrarily small. Consequently, on a coarse scale there is no vorticity and therefore no flow. This configuration was anticipated by Kelvin (1887) and dubbed the 'vortex sponge'. However, this perhaps uninteresting flow is not necessarily the outcome in cases where the net vorticity in the domain is non-zero. In this instance it is possible that a vortex sponge which has uniform, non-zero coarse-grained vorticity will now have more energy than the initial condition. Consequently modified dynamics with decreasing energy must evolve to a different state. One unambiguous example is where the domain contains only fluid with positive or zero vorticity. Thinking of the mechanical analogy of a membrane clamped at its edge, in which stream function is displacement and vorticity is load, it is clear that the minimum-energy arrangement of the load is achieved by moving it as close as possible to the edge, so that the support bears its weight. It is also clear that the maximum-energy configuration is found by placing the load in the middle of the membrane, as far from the edge as possible.

Since the final state is obtained by advecting the initial vorticity, it is both dynamically and topologically accessible (in the sense of Moffatt 1985) from the initial state. Flows which lie on the same sheet have similar topological properties of the vorticity field. Consider a simple two-dimensional case, in which the initial vorticity field is piecewise continuous, being composed of 'vortex patches' of uniform vorticity (figure 5). Any subsequent evolution of the flow must conserve not only the area of all the patches, but also the structure of the linkages between them. Thus, an evolution in finite time to a state illustrated in figure 5(*b*) is not allowed, even though the integral of all vorticity invariants may be maintained. On the other hand, evolution to the state of figure 5(*c*) is allowed. However, the final state need not be

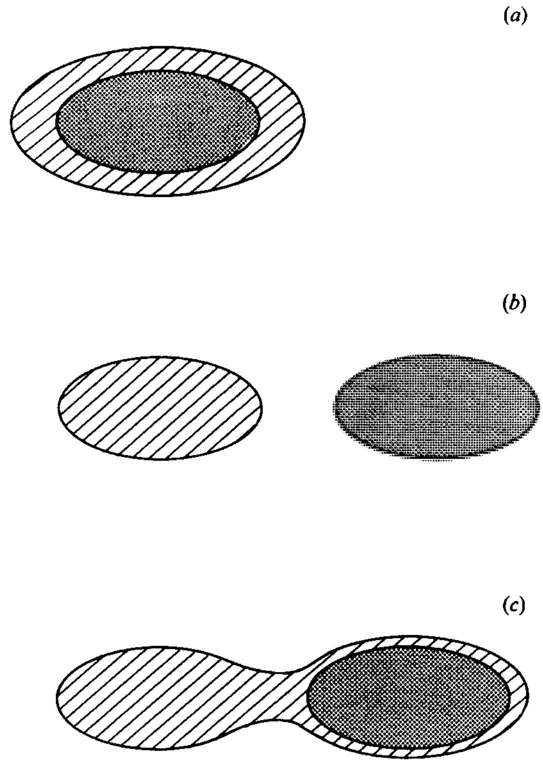

FIGURE 5. Schematic evolution of vortex patches. The areas of the patches remain conserved. (*a*) Initial state, (*b*) forbidden (in finite time) later state, (*c*) allowed later state.

topologically equivalent to the initial state, if by topological equivalence we mean obtainable by a homeomorphism. This is because at $t = \infty$ the final state may not be continuous (even in a piecewise sense). As an illustrative example (with no dynamical relevance necessarily intended) consider the initial state of two vortex blobs of the same sign, with zero vorticity between (see figure 6). Let us suppose that the blobs draw closer and attempt to merge. A dynamically possible and topologically equivalent later state is shown in figure 6 (*b*). As time progresses the two patches may become more intertwined, but since vorticity is conserved on parcels the entire space remains three-valued. At infinite time, the vortex values will not merge to produce a single blob of vorticity $\omega_3 = \frac{1}{2}(\omega_1 + \omega_2)$. Rather, the blobs become infinitely intertwined; any coarse-grained view of the vorticity would see only the average vorticity, but a finer view would reveal infinitely thin filaments (with finite total measure) of values ω_1, ω_2 and possibly zero, only.

Of course in many situations (to be reported on in a subsequent paper) the modified dynamics will in finite time evolve to a stable state which will be as continuous as the initial state and strictly topologically equivalent. In all cases it is dynamically and topologically accessible from the initial state.

6.2. *Three-dimensional flow*

It is a widely received opinion that in three dimensions no stable solutions of the Euler equations with non-trivial topology exist. Arnol'd (1965 *b*) remarks that he was unable to find a flow $\boldsymbol{u}$ for which the second variation of the energy to isovortical

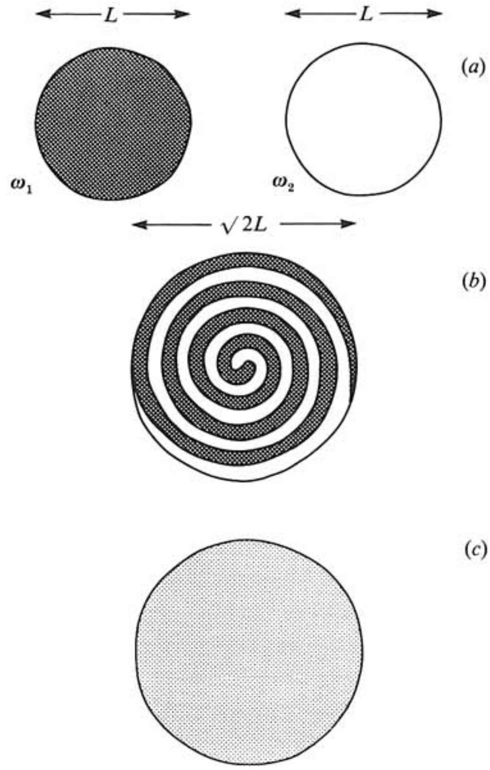

FIGURE 6. Schematic merger of two vortex blobs. (a) Initial configuration, $t = 0$, (b) state at later time $0 < t < \infty$, (c) state at $t = \infty$.

perturbations is of fixed sign, which amounts to the same thing. The arguments in the introduction imply that a stable solution would exist if either an upper or lower bound to the energy can be found, while maintaining the vorticity invariants.

However, this does not seem possible, principally because it is impossible to bound vorticity from above. To see this, consider the Schwartz inequality:

$$\int u^2 \,\mathrm{d}V \int \omega^2 \,\mathrm{d}V \geqslant \left[\int \boldsymbol{u} \cdot \boldsymbol{\omega} \,\mathrm{d}V \right]. \tag{6.2}$$

Noting that the right-hand side is constant we see that a lower bound on the energy would follow from an upper bound on the enstrophy (and vice versa). However, just as in conventional hydrodynamics, an enstrophy bound is not forthcoming. If the energy increases, the helicity invariant makes the velocity and vorticity fields everywhere orthogonal. The case where energy decreases seems more hopeful, since the velocity field can only not be bounded if vorticity becomes infinite; it seems counter-intuitive that vorticity should go to infinity as energy decreases. Nonetheless, this happens if material lines are stretched indefinitely as the area they enclose shrinks without limit. In this way the circulation $\oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{r}$ is conserved because the path becomes infinite (even though $\boldsymbol{u} \rightarrow 0$). Additionally, $\int \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}$ is conserved as $\boldsymbol{\omega} \rightarrow \infty$ because the area shrinks. Thus to bring the energy to zero the modified dynamics may continuously fold the vortex lines back on themselves, producing bundles in which anti-parallel filaments are juxtaposed, and the smoothing effect of the Biot–Savart law results in small velocities. This is a three-dimensional analogue of Kelvin's vortex sponge.

Interestingly we note that, as the energy decreases with constant helicity, we would normally expect the flow to become increasingly 'Beltramized', with $\boldsymbol{u}$ parallel to $\boldsymbol{\omega}$. This is because a solution of the variational problem of extremizing energy with helicity fixed gives Beltrami flow. Explicitly, solving $\delta \int (u^2 - \lambda \boldsymbol{u} \cdot \boldsymbol{\omega})\, \mathrm{d}V = 0$, yields $\boldsymbol{u} = \lambda \boldsymbol{\omega}$. Recent numerical simulations of magnetohydrodynamic flow by Dahlburg *et al.* (1987) indicate that the cascade of energy to small scales may not be accompanied by a similar cascade of magnetic helicity. (The simulations have insufficient resolution to resolve an inertial range, so actually one can only say that energy decays more rapidly than topological invariants like helicity.) Similar phenomena are likely in neutral flows. Thus the large scales of a turbulent flow maintain their initial helicity, and are presumably organized into coherent Beltrami-like structures. This is an example of 'selective decay' (Matthaeus & Montgomery 1980). Now, whereas minimizing energy with fixed helicity certainly leads directly to the Beltrami condition, other more complicated topological invariants of the vorticity field presumably also remain trapped in the large scales, preventing the complete realization of a Beltrami state. Incorporating these constraints directly into a variational problem seems hopelessly complicated. Modified dynamics with decreasing energy, on the other hand, seems an effective method of equivalently solving the problem. Further, in some sense it may be thought of as a parameterization of the cascade of energy to small unresolved scales in three-dimensional neutral and magnetohydrodynamic turbulence. An analogous procedure, successful in two dimensions, is the 'anticipated potential vorticity' method of Sadourny & Basdevant (1985). Here it is the energy that remains at large scales while the enstrophy cascades to small, and their parameterization models this by conserving energy while dissipating enstrophy.

7. Discussion

In this paper we have considered possible ways of finding stable solutions of the Euler equations. Stability in Euler flows is assured for states that are conditional extrema of energy subject to an isovortical rearrangement of the fluid. Our strategy has been to modify the dynamics in order to preserve the vorticity invariants while systematically changing the energy.

Simple inequalities tell us that energy is bounded from above if vorticity is bounded. In fact vorticity is bounded in two-dimensional flow, but not in three. Thus in two dimensions energy extrema of the Euler equations exist for any configuration of the vorticity field. Further, we have presented a method which, while the motion is unsteady, smoothly and isovortically maps a flow to another of higher or lower energy. Thus the end state must be either a stable solution or a singular extremum of the equations. The dynamics therefore provide a practical method for searching for stable solutions of the equations of motion.

The method appears sufficiently general that application in other fields seems likely. Consider, for example, any dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\{x_i\}), \tag{7.1}$$

where $\boldsymbol{x}$ is a multi-dimensional state vector, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. Suppose the system has a conserved quantity Σx_i^2, which we call energy, obtained by taking the dot product of (7.1) with $\boldsymbol{x}$. Then to form a modified set of dynamics for the system (7.1) we can replace the ith component of $\boldsymbol{x}$, namely x_i, by $x_i + \alpha \dot{x}_i$ in all the right-hand sides of the equations. Then, in general, the energy of the system will change monotonically by $\alpha \dot{x}_i^2$. The trick, of course, is to choose the form of the equations and the variable in such a way that other invariants remain preserved, so that the solution may evolve to a stable steady state.

A number of issues demand attention for future work. Perhaps the most pressing is to explore the numerical implementation of these dynamics in two and three dimensions, to see if both new and old solutions can be found. In two dimensions we have implemented the procedure with many interesting results; these will be reported on in a subsequent paper. Another endeavour will be to use the method as a tool to investigate the existence of stable solutions, particularly in three dimensions where it is believed by many that none exist. Thus, for example, it may be useful to explore, analytically and numerically, application of the method to given, unsteady, initial states. How would the Taylor–Green vortex evolve under the modified dynamics? Is it possible to obtain *any* non-trivial stable solutions?

We have also speculated that modified dynamics may have physical significance in emulating some of the transfer properties of the inertial range in three-dimensional turbulence. There are some indications that three-dimensional turbulence is characterized by a cascade of energy to small scales, while the helicity remains at the large scales. The upshot of this would be a tendency towards 'Beltramization' of the large eddies, with a subsequently inhibited energy transfer. Because its effects are similar, modified dynamics could be used as a means of parameterizing the action of small, unresolved scales on those explicitly resolved in a turbulence calculation. Of course, indications as to whether such a scheme is better or worse than a more conventional parameterization are most likely to come, in the absence of any compelling theory, from numerical experimentation.

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